Mathematical Modeling and Study of Oscillation of Lotka-Volterra by Jacobian Matrix Method

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Abstract— Lotka Volterra investigates a wide range of ecological problems. It includes the effects of migration and simultaneous interaction of several species. It is used to characterize predator-prey interaction.

This paper investigated the thermodynamic stability and mathematical modeling of Lotka-Volterra model proceeding at finite rate by using the Jacobian matrix method. The stability analysis of Lotka-Volterra model is complex in mechanism. The feedback mechanism of this model also leads to the oscillatory chemical reaction.

Keywords— Irreversible Thermodynamics, Stability, Lotka–Volterra model, Jacobian matrix.

I. INTRODUCTION

The Jacobian matrix method¹-⁵ is an important tool to analyze the effect of the perturbation on the steady state. This method is best suited for the linear systems where the marginal stability have an unusual importance in which the system parameter changes and a state goes from being stable to unstable or vice-versa. In a system with single concentration variable the stability analysis is obviously easier to deal with, both algebraically and intuitively. However, in the real system where the several dynamic variables are co-exist which are sensitive to external perturbation, do not deal with a normal algebra. The periodic oscillations require at least two independently varying concentrations. In such cases, the efforts required to analyze a model goes up very rapidly as the number of concentration variables increases. For examples, the Lotka-Volterra ecological model³-⁸ is two-variable system while Oregonator model (Noyes and Field, 1974)³-⁶,⁹,¹¹ for the BZ reaction is three variables system. In these systems, periodic oscillation, structures and order in chaos are the important phenomena. It is great deal to analyze the stability of such systems. So, Jacobian is the important tool to deal several parallel equations simultaneous to scrutinize the stability and instability of the systems.

One very important mathematical result facilitates the analysis of two-dimensional systems. The Poincare-Bendixon¹-⁶ theorem states that, if two dimensional system is confined to a finite region of concentration space then it must ultimately reach to a steady state or oscillate periodically. The system cannot move through the concentration space indefinitely; the only possible asymptotic solution, other than the steady state, is oscillations. This result is extremely powerful, but it holds only for two dimensional systems. Thus if a two-dimensional system has no stable steady state and that all concentrations are bounded, that is, the system cannot explode then that the system has stable periodic solution, without identification of an exact solution.

Let us consider a model with two independent concentrations, α and β, whose time derivatives are represented by two functions f and g respectively:

\[
\frac{d\alpha}{dt} = f(\alpha, \beta) \tag{1}
\]
\[
\frac{d\beta}{dt} = g(\alpha, \beta) \tag{2}
\]

The steady state concentrations \((\alpha_s, \beta_s)\), are now perturbed by small elements, \(\delta\alpha\) and \(\delta\beta\) respectively. The perturbed state in this case represent as

\[
\alpha = \alpha_s + \delta\alpha, \tag{3}
\]
\[
\beta = \beta_s + \delta\beta. \tag{4}
\]

On substituting the eqs.(3) and (4) into eqs.(1) and (2) and expand the function of \(f\) and \(g\) in the Taylor series about
the steady state points \((\alpha_n, \beta_n)\), where \(f = g = 0\). If the perturbations are small enough, the second and higher order terms are insignificant and equation becomes,
\[
\frac{d(\delta \alpha)}{dt} = (\delta f / \delta \alpha)_n \delta \alpha + (\delta f / \delta \beta)_n \delta \beta, \tag{5}
\]
\[
\frac{d(\delta \beta)}{dt} = (\delta g / \delta \alpha)_n \delta \alpha + (\delta g / \delta \beta)_n \delta \beta. \tag{6}
\]
Eqs(5) and (6) are just the differential equation in \(\delta \alpha\) and \(\delta \beta\). Equations of this form have solutions that are the sums of exponentials, where the exponents are found by assuming that each variable is of the form \(c_i \exp(\lambda t)\). Using these facts let us define
\[
\delta \alpha(t) = c_1 e^{\lambda t}, \quad \delta \beta(t) = c_2 e^{\lambda t} \tag{7}
\]
where \(c_1\) and \(c_2\) are the vector coefficients corresponds to respectively \(\delta \alpha\) and \(\delta \beta\), \(\lambda\) is the eigenvalue (undetermined multiplier). Further, the Jacobian matrix \(J\) is usually defined as,
\[
J = \begin{bmatrix}
\frac{\partial f}{\partial \alpha} & \frac{\partial f}{\partial \beta} \\
\frac{\partial g}{\partial \alpha} & \frac{\partial g}{\partial \beta}
\end{bmatrix}
\tag{8}
\]

The values of elements of \(J\) are obtained from eqs.(5) and (6). The results obtained on substituting eq.(7) into eqs.(5) and (6) and dividing by \(\exp(\lambda t)\) can be written in the compact form as,
\[
(J - \lambda I)C = 0 \tag{9}
\]
where \(J\) is the Jacobian matrix defined in eq.(8) and \(C\) is the vector coefficients \((c_1, c_2)\). \(I\) is the \(2 \times 2\) identity matrix and 0 is a \(2 \times 1\) vector of zeros. Eq.(9) have the non trivial solution-that is the solutions other than all the coefficient \(c\) being zero-only when \(\lambda\) is a solution of the determinantal equation
\[
\det \begin{bmatrix}
\frac{\partial f}{\partial \alpha} - \lambda & \frac{\partial f}{\partial \beta} \\
\frac{\partial g}{\partial \alpha} & \frac{\partial g}{\partial \beta} - \lambda
\end{bmatrix}_n
= \left(\frac{\partial f}{\partial \alpha} - \lambda\right) \left(\frac{\partial g}{\partial \beta} - \lambda\right) - \left(\frac{\partial g}{\partial \alpha}\right)_n \left(\frac{\partial f}{\partial \beta}\right)_n = 0 \tag{10}
\]
On expanding eq(10), gives
\[
\lambda^2 - \lambda tr(J) + \det(J) = 0 \tag{11}
\]
where \(tr(J)\) is the trace of the Jacobian matrix. Eq(11) is the quadratic in the exponent \(\lambda\), which has two solutions and whose coefficients depends on the elements of Jacobian, that is, the steady state concentrations and the rate constants. The general solution to eqs.(10) and (11) for the time evolution of the perturbation will be a linear combinations of two exponentials. The stability of the system will be determined by whether the perturbation grows or decays. If either eigenvalue \(\lambda\) has a positive real part, the solution will grow and the steady state is unstable. If both \(\lambda\) values have negative real part, the steady state is stable.

The above analysis can be generalized to systems with any number of variables. If we write the rate equations as
\[
\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n), \quad (i = 1, 2, \ldots, n) \tag{12}
\]
We can define the elements of \(n \times n\) Jacobian matrix associated with the steady state \(X_n\) as
\[
J_{ij} = \left(\frac{\partial f_i}{\partial x_j}\right)_{ss} \tag{13}
\]

The steady state will be stable if all the eigenvalues of \(J\) have negative real parts. If any of them has positive real parts, the state is unstable.

The behavior of two-dimensional systems based on the nature of the solution to eq(11) and are obtained by applying the quadratic formula. We will now consider the several possibilities for the sign of the trace, determinant, and discriminant, \((tr(J)^2 - 4 det(J))\) identified by Gray and Scott\(^6\) to assign the stability and instability of the processes:

(a)
\[
tr(J) < 0, \quad det(J) > 0, \quad tr(J)^2 - 4 det(J) > 0
\]
If these inequalities hold, then both eigenvalues are negative real numbers. Any perturbation to this steady state will monotonically decrease and disappear. The steady state and nearby points in the concentration space are the stable.

(b)
\[
tr(J) < 0, \quad det(J) > 0, \quad tr(J)^2 - 4 det(J) < 0
\]
The last inequality implies that the eigenvalues will be complex conjugates, that is of the form \(\lambda = a \pm ib\). The real part of both the eigenvalues are negative, meaning that the perturbation will decay back to the steady state. The imaginary exponential is equivalent to a sine or a cosine function, which implies that the perturbation will oscillate as it decays. This steady state is called a stable focus.
III. STEADY STATE ANALYSIS

In steady state the concentrations of intermediate species X and Y remain time independent, that is

\[ \frac{d[X]}{dt} = k_1[A][X]^2 - k_2[X]^2[Y]^2 = v_1^s - v_1^f = 0 \quad (14) \]

\[ \frac{d[Y]}{dt} = k_2[X]^2[Y]^2 - k_3[Y]^3 = v_2^f - v_2^s = 0. \quad (15) \]

where \([X]^s\) and \([Y]^f\) are the concentrations of X and Y in steady state respectively. Notice that the eqs. (14) and (15) gave the new identity that is

\[ V_1^f = V_2^s = V_3^s = V. \quad (16) \]

After solving eqs. (14) and (15), the steady state concentrations of X and Y are obtained as

\[ [X]^s = \left( \frac{k_3}{k_1[A]} \right) [Y]^s \quad [X]^f = \left( \frac{k_3}{k_2} \right) [Y]^f = \left( \frac{k_3}{k_3} \right) A. \quad (17) \]

The stability of steady state has been investigated by Jacobian matrix and Lyapunov direct method of stability of motion at constant \( T \) and \( p \) in subsequent sections.

IV. STABILITY ANALYSIS BY JACOBIAN MATRIX METHOD

For the treatment of stability by Jacobian matrix method, we assume that the rate constants equal to unity, \([A] = a, [Y]^f = y \) and \([X]^f = x.\) Under these assumptions, the rate expressions for X and Y respectively read as

\[ \frac{dx}{dt} = ax - xy = 0 \quad (18) \]

\[ \frac{dy}{dt} = xy - y = 0. \quad (19) \]

Then, the steady state concentrations of X and Y become

\[ [Y]^s = y = a, \quad [X]^f = x = 1 \quad (20) \]

To analyze the stability of the state, we must calculate the elements of Jacobian matrix, \( J; \)
\[ J = \begin{bmatrix} \frac{\partial (dx/dt)}{\partial x} & \frac{\partial (dx/dt)}{\partial y} \\ \frac{\partial (dy/dt)}{\partial x} & \frac{\partial (dy/dt)}{\partial y} \end{bmatrix} \] \quad (21)

The elements of Jacobian matrix are obtained as

\[
\frac{\partial (dx/dt)}{\partial x} = a - y = a - a = 0, \quad (22)
\]

\[
\frac{\partial (dx/dt)}{\partial y} = x = -1 \quad (23)
\]

\[
\frac{\partial (dy/dt)}{\partial x} = y - 0 = a \quad (24)
\]

\[
\frac{\partial (dy/dt)}{\partial y} = x - 1 = 1 - 1 = 0 \quad (25)
\]

Thus, on substituting values of gradients from eqs.(22)-(25) in matrix \( J \), we obtain

\[
J = \begin{bmatrix} 0 & -1 \\ a & 0 \end{bmatrix} \quad (26)
\]

We now need to obtain the eigenvalues of the matrix whose elements are given by eqs. (22)-(25) by solving the characteristic equation:

\[
\det \begin{bmatrix} 0 - \lambda & -1 \\ a & 0 - \lambda \end{bmatrix} = 0 \quad (27)
\]

or equivalently

\[
\lambda^2 + a = 0 \implies \lambda = i\sqrt{a} \quad (28)
\]

Similarly, from Jacobian matrix, eq.(26), we obtain \( \text{tr}(J) = 0 \),

\( \det(J) = a > 0 \)

\( \text{tr}(J)^2 - 4\det(J) < 0 \quad (29) \)

It is observed that when the determinant is positive and the trace is zero, the eigenvalue become purely imaginary. Such condition indicates the onset of sustained oscillation through a Hopf bifurcation\(^{(10,13)}\). As a system parameter is varied so that the system passes through the Hopf bifurcation limit cycle or periodic orbit develops surrounding the steady state. If the limit cycle is stable in which case the bifurcation is said to be supercritical, the steady state loses its stability and any small perturbation will cause the systems to evolve into a state of sustained oscillations. In a subcritical Hopf bifurcation, the steady state maintains its stability, but becomes surrounded by a pair of limit cycles, the inner one being unstable and the outer one being stable. Such a situation gives rise to bistability between steady state and the stable oscillatory state. It is a periodic oscillation on its own and any perturbation can drive the oscillations into a different cycle.

The result derived from software are given below:

1) There is domain in periodic oscillations which gives clear instability on perturbation of intermediate coordinates. For example, when perturbation is executed before 1st s of onset of oscillations, the analysis shows that the process is unstable. The behaviour of \( \delta X, \delta Y \), with time, are shown in Fig 1. Note that the graphical presentation of \( \delta X, \delta Y \), shows they have same (positive) signs. This indicates that the perturbation is not decayed rather it grows with time. Hence, small perturbation in this region is unstable and uncontrollable.

2) At certain points of chemical oscillations, perturbation leads to formation of another pattern of oscillations of coordinates. From graphical presentation of Mathematica software shows that these oscillations grow with time beyond the control shown in Fig 2 and Fig.3. This is again the clear case of instability of the process.

![Fig 1: Variation of \( \delta X \) and \( \delta Y \) with time](image)

![Fig 2: Variation of \( \delta X \) and \( \delta Y \) with time](image)

![Fig 3: Variation of \( \delta X \) and \( \delta Y \) with time](image)
Our analysis shows that the chemical oscillations in Lotka-Volterra model are very sensitive to small perturbation. There is big domain of chemical oscillations which is susceptible to instability of the process on effect of perturbation. In some domain, perturbation tends to onset of oscillatory behavior. There is very tiny domain over which the stability on perturbation is guaranteed.

V. CONCLUSION

Lotka-Volterra is the first mathematical model on ecology where periodic growth and decay of animals in certain region of space are modeled. In this chapter, two physical situation of animal world: nonequilibrium stationary state and oscillatory state have been discussed. Our investigation shows that the stationary state is very sensitive to external disturbances. The small perturbation of intermediate coordinates from their steady state concentrations generates sustained oscillation and state becomes unstable. Our stability analysis of chemical oscillations shows that the stability of state is ensured over very small domain. However, very large domain of chemical oscillation is unstable to the perturbation. The domain stability, instability and oscillatory is clearly revealed in our investigation.

REFERENCES


