Isothermal Cylindrical Couette Flow Of Oldroyd 8-Constant Model Fluid Up To Second-order

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Abstract—*The steady flow of an incompressible Oldroyd 8-constant fluid in the annular region between two concentric cylinders, or so-called cylindrical Couette flow, is investigated. The inner cylinder rotates with an angular velocity Ω about its own axis, z-axis, while the outer cylinder is kept at rest. The viscoelasticity of the fluid is assumed to dominate the inertia such that the latter can be neglected in the momentum equation. An analytical solution is obtained through the expansion of the dynamical variables in power series of the dimensionless retardation time. The primary velocity term denotes the Newtonian rotation about the z-axis. The first-order is a vanishing term. The second-order results in a secondary flow represented by the stream-function. This second-order term is the viscoelastic contribution to the primary viscous flow. The second-order approximation depends on the four viscometric parameters of the fluid.*

Keywords—cylindrical Couette flow, Oldroyd 8-constant model, secondary flow.

I. INTRODUCTION

Fluid flow in the annular region between two rotating objects has attracted the attention for the last few decades in many branches of industries and technology [1-6]. One of these flow problems is the viscous and viscoelastic flow between two cylinders. Fluids are generally classified based on their rheological properties. The simplest classification is Newtonian fluid. These fluids are represented using Navier-Stokes theory. The fluids which do not obey Newton's law of viscosity are described as non- Newtonian fluids. Examples of such fluids are blood, saliva, semen, lava, gums, slurries, emulsions, synovial fluid, butter, cheese, jam, ketchup, soup, mayonnaise etc.

Numerous models, or so called constitutive equations, have been proposed to describe the response of fluids that cannot be characterized by the classical Navier-Stokes fluid model. The progress of any constitutive equation based on the investigated rheolegical properties of a fluid behavior under consideration simplest constitutive equation for a fluid is a Newtonian one and the classical Navier-Stokes theory is based on this equation. The mechanical behavior of many fluids is well enough described by this theory. However, there are many rheologically complicated fluids such as polymer solutions, blood and certain oils, suspensions, liquid crystals in industrial processes with non-linear viscoelastic behavior that cannot be described by a Newtonian constitutive equation, as it does not reflect any relaxation and retardation phenomena.

In recent years, the interest for non-Newtonian fluid flows has considerably increased and many exact solutions have been obtained [7-10]. For this reason, many models have been proposed in which fluids are usually classified as of differential rate and integral types [11]. The differential type models are used to describe the response of the fluids which have slight memory such as dilute polymeric solutions whereas the integral models are used to describe the response of the fluids which have considerable memory such as polymeric melts. A large number of non-Newtonian fluid models are concerned with the fluids of grade two and three, but these fluids do not predict stress relaxation and retardation. Models of the Maxwell, Oldroyd-B and Burgers'fluid type can predict these phenomena, and have therefore become more popular. The Oldroyd-B fluid model [12] which takes into account elastic and memory effects exhibited by most polymeric and biological liquids, has been used quite widely in many applications and the results of simulations to experimental data in a wide range.[13] The Oldroyd-B fluids belong to the class of rate type fluid models[14-22]. Generally, there exist three kinds of boundary value problems: (1) when the velocity is given on the boundary; (2) when the shear stress is given on the boundary and (3) mixed boundary value problems. From the applications point of view, the first one of boundary value problems are of interest. Axial flow in an annulus between a rotating inner cylinder and a fixed outer cylinder has several important engineering applications including journal bearings, biological separation devices, rotating machinery, desalination to magneto hydrodynamics and also in viscosimetric analysis[23-25].

At present, the flow field of an incompressible Oldroyd 8-costant fluid in the annular region between two concentric cylinders is investigated. The inner cylinder rotates with a uniform velocity Ω about its own z-axis centered in the origin of the system and the outer cylinder is being at rest. The suggested Oldroyd 8-costant constitutive equation is linear in the stresses alone and contains all possible terms quadratic in the stress and the velocity-gradient components consistent with giving a symmetric stress tensor. Moreover, this model already presents a considerable simplification with respect to general models of simple fluids. Hence, this model can describe more rheological behaviors method of solution and the obtained results.

II. FORMULATION OF PROBLEM

A viscoelastic fluid moves in the annular space between two concentric cylinders of radii R_1 and R_2 ($R_2 > R_1$). The motion is due to the rotation of the inner cylinder with angular velocity Ω about its axis , z-axis, while the outer cylinder is kept at rest.

2.1 DIMENSIONLESS VARIABLE

In the present work, for the dimensional variable-quantities; namely, the length $\tilde{r}$, velocity $\underline{v}$, deformation tensor $\underline{\underline{d}}$, stress tensor $\underline{\underline{t}}$, pressure p and the stream-function ψ , it is more convenient to introduce the following dimensionless variables in terms of non-dimensional cylindrical polar coordinates (r,θ,z) ,

$$\left[\begin{array}{l} r=\tilde{r}/R_1;\ \underline{V}=\underline{v}/\Omega R_1=(U\hat{r},V\hat{\theta},W\hat{\varphi}); \\ \underline{\underline{D}}=\underline{\underline{d}}/\Omega;\ P=p/\eta_0\Omega,\ \Psi=\psi/R_1^3\Omega,\underline{\underline{T}}=\underline{\underline{t}}/\eta_0\Omega \end{array}\right] \quad (2\text{-}1)$$

where, the non-dimensionality is obtained by using R_1, Ω and $\eta_0\Omega$ as the characteristic length, time and stress; respectively.

2.2 CONSTITUTIVE EQUATION

To get a comprehensive idea about the behavior of a viscoelastic fluid, we adopted the Oldroyd 8-costant model in the present work. This model represents one of the most general constitutive equations for the last four decades, Zmievski et al.[26], and it includes a number of frequently used constitutive equations as special cases [27,28,29]; (see Table (A-1) of appendix A). Here, we shall use Oldroyd's suggestion [27,30,31] which reduces the model to 6-parameters model by assuming that $\lambda_3 = \lambda_4 = 0$

Oldroyd constitutive equation relates the stresses and the kinematic variables through the non-dimensional equation; [30],

$$\underline{\underline{T}}=2\underline{\underline{D}}-\lambda\left[\xi_1\overset{\nabla}{\underline{\underline{T}}}+\xi_5(tr\underline{\underline{T}})\underline{\underline{D}}-2\overset{\nabla}{\underline{\underline{D}}}+(\xi_6\underline{\underline{T}}:\underline{\underline{D}}-2\xi_7\underline{\underline{D}}:\underline{\underline{D}})\underline{\underline{I}}\right], \quad (2\text{-}2a)$$

and

$$\underline{\underline{T}}_{exc} = -P\underline{\underline{I}} + \underline{\underline{T}} , \quad (2\text{-}2b)$$

$$\lambda=\lambda_2\Omega; \qquad \xi_i=\frac{\lambda_i}{\lambda_2} \quad for\ i=1,5,6,7 ,$$

where $\underline{\underline{T}}$, $\underline{\underline{D}}=\frac{1}{2}[(\nabla\underline{V})+(\nabla\underline{V})^T]$ and $\underline{\underline{I}}$ are the extra-stress, the rate of deformation and the unit tensors; respectively. The material constants η_o , λ_1 and λ_2 are the viscosity, the relaxation and retardation times ; respectively, while λ_3 … λ_7 are further material time constants , $\underline{\underline{T}}_{exc}$ is the Cauchy stress tensor. The upper-convected derivative for any symmetric tensor $\underline{\underline{G}}$ is defined by

$$\overset{\nabla}{\underline{\underline{G}}}=\frac{\partial \underline{\underline{G}}}{\partial t}+\underline{V}.\nabla\underline{\underline{G}}-\underline{\underline{G}}\cdot\nabla\underline{V}-(\underline{\underline{G}}\cdot\nabla\underline{V})^T .$$

2.3 THE CONTINUITY AND MOMENTUM EQUATION

The dimensionless velocity field may be written as

$$\underline{V} = [(U(r)\hat{r},V(r)\hat{\theta},W(r)\hat{z}] . \quad (2\text{-}3)$$

The continuity equation is satisfied identically by introducing a stream-function defined by

$$\underline{U}_\perp = U\,\hat{r} + V\hat{\theta} = -\nabla\wedge(\psi\hat{z}) . \quad (2\text{-}4)$$

We assume that the forces due to viscoelasticity are dominated such that the inertial term $\underline{V}\cdot\nabla\underline{V}$ is negligible (2-1) in the momentum equation. Thus, for steady state the momentum equation; $-\nabla P+\nabla\cdot\underline{\underline{T}}=0$, is given by

$$-\nabla P+2\nabla\cdot\underline{\underline{D}}-\lambda\left\{\nabla\cdot\left[\xi_1\overset{\nabla}{\underline{\underline{T}}}+\xi_5(tr\underline{\underline{T}})\underline{\underline{D}}-2\overset{\nabla}{\underline{\underline{D}}}\right]-\nabla\left[\xi_6\underline{\underline{T}}:\underline{\underline{D}}-2\xi_7\underline{\underline{D}}:\underline{\underline{D}}\right]\right\}=0, \quad (2\text{-}5)$$

The perturbation term, Eq. (2-2b), is abbreviated in Eq. (2-5) by the vector $\underline{\Lambda}(r,\theta)$ and the scalar $G(r,\theta)$ defined by the expressions

$$\underline{\Lambda}(r,\theta) = \nabla\cdot\left[\xi_1\overset{\nabla}{\underline{\underline{T}}}+\xi_5(tr\,\underline{\underline{T}})\underline{\underline{D}}-2\,\overset{\nabla}{\underline{\underline{D}}}\right] \quad (2\text{-}6a)$$

$$G(r,\theta)=\left\lfloor \xi_6\underline{\underline{T}}:\underline{\underline{D}}-2\xi_7\underline{\underline{D}}:\underline{\underline{D}}\right\rfloor, \quad (2\text{-}6b)$$

Hence, the momentum equation, Eq.(2-5), can be factorized into the pair of equations

$$\text{(i)}\ \nabla^2(W\hat{z})-\lambda\,\Lambda_3\hat{z}=0 , \quad \Lambda_3=\hat{z}\cdot\underline{\Lambda} , \quad (2\text{-}7)$$

Which reduces to the scalar equation

$$\frac{1}{r}\left[\partial_r(rW_{,r})+\frac{1}{r}\partial^2{}_\theta(W)\right]-\lambda\Lambda_3=0 . \quad (2\text{-}8)$$

$$\text{(ii)}\ \nabla^2\underline{U}_\perp-\lambda\underline{\Lambda}_\perp-\nabla(P-G)=0 ;\ \underline{\Lambda}_\perp=\underline{\Lambda}-\hat{z}\Lambda_3 , \quad (2\text{-}9)$$

Taking the curl of Eq.(2-9) and substituting from Eq.((2-4), we get the scalar equation

$$\left[\frac{1}{r}\partial_r(r\partial_r)+\frac{1}{r^2}\partial_\theta^2\right]^2\Psi-\frac{\lambda}{r}[\partial_r(r\Lambda_2)-\partial_\theta\Lambda_1]=0, \quad (2\text{-}10)$$

Where,

$$\underline{\Lambda}_\perp=\Lambda_1\hat{r}+\Lambda_2\hat{\theta}\ . \quad (2\text{-}11)$$

The components of $\underline{\Lambda}$ are given as:

$$\Lambda_1=\hat{r}.\underline{\Lambda}=r^{-1}\partial_r[r(\xi_1\overset{\nabla}{T}_{rr}-2\overset{\nabla}{D}_{rr}+\xi_5\Gamma D_{rr})]+r^{-1}\partial_\theta[r(\xi_1\overset{\nabla}{T}_{r\theta}-2\overset{\nabla}{D}_{r\theta}+\xi_5\Gamma D_{r\theta})]+\frac{1}{r}[2\overset{\nabla}{D}_{\theta\theta}-\xi_1\overset{\nabla}{T}_{\theta\theta}-\xi_5\Gamma D_{\theta\theta}] \quad (2\text{-}12a)$$

$$\Lambda_2=\hat{\theta}.\underline{\Lambda}=r^{-2}\partial_r[r^2(\xi_1\overset{\nabla}{T}_{r\theta}-2\overset{\nabla}{D}_{r\theta}+\xi_5\Gamma D_{r\theta})]+r^{-1}\partial_\theta[r^2(\xi_1\overset{\nabla}{T}_{\theta r\theta}-2\overset{\nabla}{D}_{\theta\theta}+\xi_5\Gamma D_{\theta\theta})] \quad (2\text{-}12b)$$

and

$$\Lambda_3=\hat{z}.\underline{\Lambda}=r^{-1}\partial_r[r\ (\xi_1\overset{\nabla}{T}_{rz}-2\overset{\nabla}{D}_{rz}++\xi_5\Gamma D_{rz})]+r^{-1}\partial_\theta[r\ (\xi_1\overset{\nabla}{T}_{\theta z}-2\overset{\nabla}{D}_{\theta rz}++\xi_5\Gamma D_{\theta z})] \quad (2\text{-}12c)$$

Where, $\Gamma=T_{rr}+T_{\theta\theta}+T_{zz}=tr(\underline{\underline{T}})$.

The components of the tensors $\underline{\underline{T}}$, $\overset{\nabla}{\underline{\underline{T}}}$, $\underline{\underline{D}}$ and $\overset{\nabla}{\underline{\underline{D}}}$ are given in appendix (A).

2.4 BOUNDARY CONDITIONS

The boundary conditions imposed on the functions W and Ψ are:

$$W=0,\quad 0 \quad \text{at} \quad r=1, a; \quad a=R_2/R_1, \quad (2\text{-}13a)$$

$$\Psi=Q,\quad 0 \quad \text{at} \quad r=1,\ a. \quad (2\text{-}13b)$$

$$\Psi_{,r}=1,\quad 0 \quad \text{at} \quad r=1,\ a. \quad (2\text{-}13c)$$

where Q is the flow rate.Then, using Eqs. (2-8) and (2-10), the functions W and Ψ are determined.

III. METHOD OFSOLUTION

Expanding the functions $W,\Psi,\underline{\underline{D}},\underline{\underline{T}}$ and P in power series of the parameter λ as follows

$$A=\sum_{n=0}\lambda^n A^{(n)}\ , \quad (3\text{-}1)$$

where A may represents any of the above functions.

Consequently, the expansion of Eqs.(2-4), (2-8)and (2-10) are:

$$\sum_{n=0}\lambda^n\left[\underline{\underline{T}}^{(n)}-2\underline{\underline{D}}^{(n)}+\lambda\sum_{k\le n}[\xi_1\overset{\nabla}{\underline{\underline{T}}}{}^{(n-k,k)}-2\overset{\nabla}{\underline{\underline{D}}}{}^{(n-k,k)}+\xi_5(tr\underline{\underline{T}}^{(n-k)})\underline{\underline{D}}^{(k)}+(\xi_6\underline{\underline{T}}^{(n-k)}:\underline{\underline{D}}^{(k)}-2\xi_7\underline{\underline{D}}^{(n-k)}:\underline{\underline{D}}^{(k)})\underline{\underline{I}}]\right]=0 \quad (3\text{-}2)$$

$$\sum_{n=0}\lambda^n\left[\frac{1}{r}[\partial_r(r\ W_{,r}^{(n)})]+\frac{1}{r^2}[\partial^2\theta(W^{(n)})]-\lambda\sum_{k\le n}\Lambda_3^{(n-k,k)}\right]=0, \quad (3\text{-}3)$$

and

$$\sum_{n=0}\lambda^n\left[\begin{array}{l}[\partial_r^2+\frac{1}{r}\partial_r+(\frac{1}{r^2}\partial^2{}_\theta)]^2\Psi^{(n)}\\ -\lambda\sum_{k\le n}r^{-1}[\partial_r(r\Lambda_2^{(n-k,k)})-\partial_\theta\Lambda_1^{(n-k,k)}]\end{array}\right]=0 \quad (3\text{-}4)$$

Notice: The expansion parameter; i.e. $\lambda=\lambda_2\Omega$, is much smaller than unity for the range of $\Omega\approx10^2$ since λ_2 is of the order ''10^{-2} to 10^{-4}'' sec. according to the values quoted in the literature; [31-32].

IV. LUTION OF THE SUCCESSIVE SET OF EQUATIONS

4.1 ZERO-ORDER APPROXIMAION

Taking $n=0$, the coefficients of λ^0 in Eqs.(3-3) and (3-4) are :

$$\left[\frac{1}{r}\partial_r(rW_{,r}^{(0)})+\frac{1}{r^2}\partial^2{}_\theta(W^{(0)})\right]=0\ , \quad (4\text{-}1)$$

and

$$[\partial_r^2+\frac{1}{r}\partial_r+(\frac{1}{r^2}\partial^2{}_\theta)]^2\Psi^{(0)}=0\ . \quad (4\text{-}2)$$

with boundary conditions

$$W^{(0)}=0,\quad 0 \quad \text{at} \quad r=1, a; \quad (4\text{-}3a)$$

$$\Psi^{(0)}=Q,\quad 0 \quad \text{at} \quad r=1,\ a. \quad (4\text{-}3b)$$

$$\Psi_{,r}^{(0)}=1,\quad 0 \quad \text{at} \quad r=1,\ a. \quad (4\text{-}3c)$$

The solution of Eq. (4-1) which satisfies boundary condition (4-3a) is

$$W^{(0)}=0. \quad (4\text{-}4)$$

The boundary conditions, Eq.(4-3b)and Eq. (4-3c) , imposed on Eq.(4-2), imply that:

$$\Psi^{(0)}=\frac{r^2}{4}(2c_2-c_3)+c_4+(c_1+\frac{r^2}{2}c_3)\ln r\ . \quad (4\text{-}5)$$

where,

$$c_3=\frac{(a+1)[2Q-1+2a^2\ln a/(a^2-1)]}{[2a(a^2-1)-((a-1)^3+4a^2))\ln a]}, \quad (4\text{-}6a)$$

$$c_2=\frac{-1}{a^2-1}-\frac{-c_3(3a-1)}{2(a+1)}), \quad (4\text{-}6b)$$

$$c_1=1-c_2\ , \quad (4\text{-}6c)$$

$$c_4=Q-\frac{c_2}{2}+\frac{c_3}{4}. \quad (4\text{-}6d)$$

Then the velocity field of zero order approximation using expansion of rlnr and taking the first two leading terms is given by

$$V=\Psi^{(0)}_{,r} \;=\frac{1}{r}(c_1+\frac{c_3}{2})+r(c_2+\frac{3}{2}c_3)-2c_3. \quad (4\text{-}7)$$

Solutions (4-4) and (4-5) and then velocity field (4-7) are stand for a Newtonian fluid.

4.2 FIRST-ORDER APPROXIMAION

Letting $n=1$, then the coefficients of λ in Eqs. (3-3) – (3-4) deliver the following pair of equations

$$\left[\frac{1}{r}\partial_r(rW^{(1)}_{,r})+\frac{1}{r^2}\partial^2{}_\theta\,(W^{(1)})\right]-\Lambda_3^{(0,0)}=0\,, \quad (4\text{-}8)$$

and

$$[\partial_r^2+\frac{1}{r}\partial_r+(\frac{1}{r^2}\partial^2{}_\theta)]^2\,\Psi^{(1)}-r^{-1}\left[\,\partial_r(r\Lambda_2^{(0,0)})-\partial_\theta\Lambda_1^{(0,0)}\right]=0. \quad (4\text{-}9)$$

Using Eqs. (2-12), the components $\Lambda_i^{(0,0)}$ *for* $i=1,2,3$ are given by

$$\Lambda_1^{(0,0)}=r^{-1}(\xi_1-2)(V_{,r}-\frac{V}{r}); \quad (4\text{-}10a)$$

$$\Lambda_2^{(0,0)}=0\,, \quad (4\text{-}10b)$$

and

$$\Lambda_3^{(0,0)}=0\,. \quad (4\text{-}10c)$$

With boundary conditions

$$W^{(1)}=0,\quad 0 \qquad \text{at} \qquad r=1,a\,; \quad (4\text{-}11a)$$

$$\Psi^{(1)}=0,\quad 0 \qquad \text{at} \qquad r=1,\; a\,. \quad (4\text{-}11b)$$

$$\Psi^{(1)}_{,r}=0,\quad 0 \qquad \text{at} \qquad r=1,\; a\,. \quad (4\text{-}11c)$$

Using Eq. (4-11a), the solution of Eq. (4-8) is,

$$W^{(1)}(r,\theta)=0 \quad (4\text{-}12)$$

The solution of Eq. (4-9) using boundary conditions, Eq. (4-11b) and Eq. (4-11c), is,

$$\Psi^{(1)}(r,\theta)=0 \quad (4\text{-}13)$$

4.3 Second-order Approximation

For $n=2$, the coefficient of λ^2 in Eqs. (3-3) and (3-4) are:

$$\left[\frac{1}{r}\partial_r(rW^{(2)}_{,r})+\frac{1}{r^2}\partial^2{}_\theta\,(W^{(2)})\right]-(\Lambda_3^{(0,1)}+\Lambda_3^{(1,0)})=0, \quad (4\text{-}14)$$

and

$$[\partial_r^2+\frac{1}{r}\partial_r+(\frac{1}{r^2}\partial^2{}_\theta)]^2\Psi^{(2)}-\left[\,\partial_r(r\Lambda_2^{(0,1)}+r\Lambda_2^{(1,0)})-\partial_\theta(\Lambda_1^{(0,1)}+\Lambda_1^{(1,0)})\right]=0. \quad (4\text{-}15)$$

Using Eq. (2-12c), the components

$$\Lambda_3^{(0,1)}=\Lambda_3^{(1,0)}=0 \quad (4\text{-}16a)$$

$$\Lambda_2^{(1,0)}=r^{-2}\partial_r[\frac{3}{8}r^2\zeta_5(\zeta_7-\zeta_6)(V_{,r}-\frac{V}{r})^3] \quad (4\text{-}16b)$$

$$\Lambda_2^{(0,1)}=r^{-2}\partial_r[\frac{-1}{4}r^2\zeta_1(\zeta_7-\zeta_6)(V_{,r}-\frac{V}{r})^3] \quad (4\text{-}16c)$$

then Eq. (4-14) becomes

$$\left[\frac{1}{r}\partial_r(rW^{(2)}_{,r})+\frac{1}{r^2}\partial^2{}_\theta\,(W^{(2)})\right]=0 \quad (4\text{-}17)$$

with the boundary conditions

$$W^{(2)}=0,0 \qquad \text{at} \qquad r=1,\; a \quad (4\text{-}18)$$

The solution of $W^{(2)}$ is,

$$W^{(2)}=0 \quad (4\text{-}19)$$

Also Eq. (4-15) becomes

$$[\partial_r^2+\frac{1}{r}\partial_r+(\frac{1}{r^2}\partial^2{}_\theta)]^2\,\Psi^{(2)}-r^{-1}\left[\,\partial_r r^{-1}(\partial_r[\frac{-1}{4}r^2\zeta_1(\zeta_7-\zeta_6)(V_{,r}-\frac{V}{r})^3+\frac{3}{8}r^2\zeta_5(\zeta_7-\zeta_6)(V_{,r}-\frac{V}{r})^3]\right]=0$$

(4-20)

which reduces to

$$[\partial_r^2+\frac{1}{r}\partial_r+(\frac{1}{r^2}\partial^2{}_\theta)]^2\,\Psi^{(2)}=24c_1^2(\frac{3}{2}\zeta_5-\zeta_1)(\zeta_7-\zeta_6)(2c_1r^{-8}-c_3r^{-6}) \quad (4\text{-}21)$$

With boundary conditions

$$\Psi^{(2)}=0,\quad 0 \qquad \text{at} \qquad r=1,\; a\,. \quad (4\text{-}22a)$$

$$\Psi^{(2)}_{,r}=0,\quad 0 \qquad \text{at} \qquad r=1,\; a\,. \quad (4\text{-}22b)$$

The solution of (4-21) with the boundary conditions (4-22a) and (4-22b) is given as:

$$\Psi^{(2)}(r)=c_1^2(\frac{3}{2}\zeta_5-\zeta_1)(\zeta_7-\zeta_6)\Psi^{*(2)}(r) \quad (4\text{-}23)$$

$$\Psi^{*(2)}(r)=(\frac{12}{145}c_1r^{-4}-\frac{4}{11}c_3r^{-2})+\Psi^h \quad (4\text{-}24)$$

where, Ψ^h is the homogeneous solution of biharmonic equation then (4-24)becomes

$$\Psi^{*(2)}(r)=(\frac{12}{145}c_1r^{-4}-\frac{4}{11}c_3r^{-2})+\frac{r^2}{4}(2B_2-B_3)+B_4+(B_1+\frac{r^2}{2}B_3)\ln r$$

(4-25)

where,

$$B_3=(1-a^2)\left[\frac{N\ln a+\frac{8}{11}c_3(\frac{a^2-1}{a^2})-\frac{36}{145}c_1(\frac{a^4-1}{a^4})}{a^2(\ln a)^2-\frac{(1-a^2)^2}{4}}\right] \quad (4\text{-}26a)$$

$$N=\frac{-8c_3}{11}(\frac{a^2+1}{a^2})+\frac{48c_1}{145}(\frac{a^4+a^2+1}{a^4}) \quad (4\text{-}26b)$$

$$B_2=\frac{8c_3}{11}(\frac{1}{a^2})-\frac{48c_1}{145}(\frac{a^2+1}{a^4})+\frac{a^2\ln aB_3}{(1-a^2)} \quad (4\text{-}26c)$$

$$B_1=-B_2-\frac{8}{11}c_3+\frac{48}{145}c_1 \quad (4\text{-}26d)$$

$$B_4 = (\frac{B_3}{4} - \frac{B_2}{2} + \frac{4c_3}{11} - \frac{12c_1}{145}) \qquad (4\text{-}26e)$$

And

$$V^{*(2)} = \Psi^{*(2)}_{,r}(r) = \left[(\frac{-48}{145} c_1 r^{-5} + \frac{8}{11} c_3 r^{-3}) + \frac{B_1}{r} + rB_2 + B_3 \ln r \right] \qquad (4\text{-}27)$$

So the second order Oldroyd velocity field is

$$V^{(2)} = \Psi^{(2)}_{,r}(r) = c_1^2 (\frac{3}{2}\zeta_5 - \zeta_1)(\zeta_7 - \zeta_6) \left[(\frac{-48}{145} c_1 r^{-5} + \frac{8}{11} c_3 r^{-3}) + \frac{B_1}{r} + rB_2 + B_3 \ln r \right] \qquad (4\text{-}28)$$

V. DISCUSSION

In the present work, the steady cylindrical Couette flow of an inertialless viscoelastic Oldroyd 8-constant fluid is investigated analytically. The flow field is created due to the rotation of the inner cylinder with radius R_1 with a uniform angular velocity Ω about its axis. The equation of momentum is solved by using the method of successive approximation through the expansion of the dynamical variables in power series of the retardation time λ. Herein, up to the second-order approximation the following results are recovered:

1- The resultant Newtonian fields streamfunction $\Psi^{(0)}(r)$ and velocity $\Psi^{(0)}_{,r}(r) = V^{(0)}$ is a solid body rotation in the θ-direction, these fields component are plotted in Fig.(1) for flow rate equal to zero and in Fig.(2)for constant value of flow rate equal to 1.

2- The first-order approximation results in a zero components in all directions.

3-The second-order approximation produces a stream function $\Psi^{(2)}(r)$ and θ-velocity component $\Psi^{(2)}_{,r}(r) = V^{(2)}$ in the direction of the primary flow. This component includes 4-parameters $\zeta_1, \zeta_5, \zeta_6 \, and \, \zeta_7$ and hence it is viscoelastic contribution.

The secondary flow represented by the stream-function $\Psi^{(2)}(r)$ and the velocity $\Psi^{(2)}_{,r}(r) = V^{(2)}$ for two cases of flow rates are plotted in Fig.(2) and Fig.(3). Since this flow is affected by six parameters; namely, η_o and λ_1 through λ_7 it is being an effect of Oldroyd 6-constant fluid.

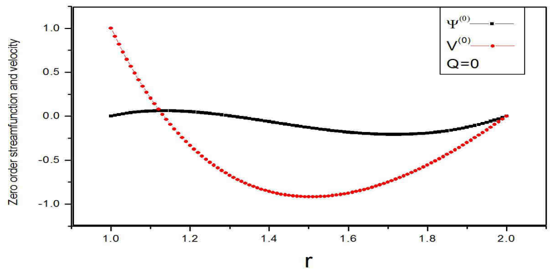

Fig.1:Zero order streamfunction and velocity field versus cylindrical radii r from inner cylinder to outer cylinder for flow rate Q=0

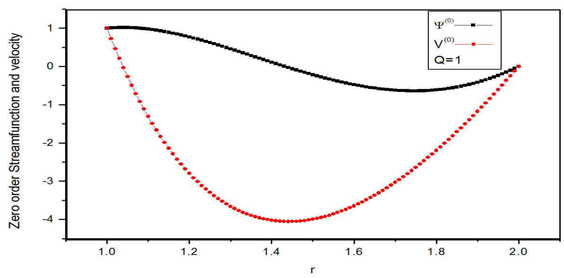

Fig.2:Zero order streamfunction and velocity field versus cylindrical radii r from inner cylinder to outer cylinder for flow rate Q=1

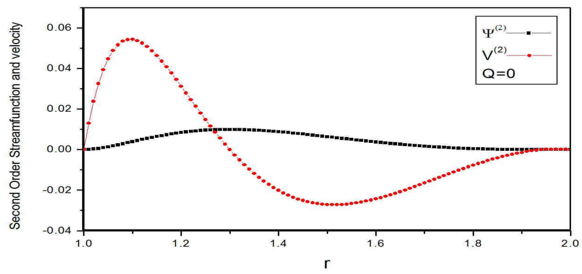

Fig.3:Second order streamfunction and velocity field versus cylindrical radii r from inner cylinder to outer cylinder for flow rate Q=0

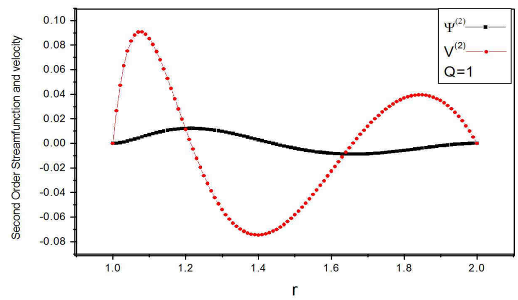

Fig.4:Second order streamfunction and velocity field versus cylindrical radii r from inner cylinder to outer cylinder for flow rate Q=1

VI. CONCLUSION

No.	Name of the model	Constants included
1-	Newtonian fluid(1-const.)	η_o ; $\lambda_i = 0$ for $i = 1,...,7$.
2-	Upper- convicted Maxwell fluid (2-const.)	η_o, λ_1 ; $\lambda_i = 0$ for $i = 2,...,7$.
3-	Oldroyd-B fluid (3-const.)	$\eta_0, \lambda_1, \lambda_2$; $\lambda_i = 0$ for $i = 3,...,7$.
4-	Fluid of second-order (3-const.)	$\eta_o, \lambda_2, \lambda_4$; $\lambda_i = 0$ for $i = 1,3,5,6,7$.
5-	GordonSchowalter or Johnson Segalman model (4-const.)	η_o ; $\lambda_2 = \frac{\eta_s}{\eta_0}\lambda_1$, $\lambda_3 = \xi\lambda_1$, $\lambda_4 = \xi\lambda_2$, $\lambda_i = 0$ for $i = 5,6,7$. η_s is the solvent viscosity.

The present work deals with the behavior of a viscoelastic fluid in the cylindrical Couette flow by using Oldroyd 8-constant model. The inner cylinder is rotating with angular velocity Ω while the outer cylinder is kept at rest. At present, the solution of the momentum equation has been performed up to the second-order. The field variables are expanded in power series in terms of the dimensionless retardation parameter λ. The zero-order velocity $V^{(0)}(r)$ is the Newtonian flow which is pure rotation about the z-axis. The second order stream-function $\Psi^{(2)}(r)$ and θ -velocity component $\Psi^{(2)}_{,r}\ (r) = V^{(2)}$ are being secondary flow which is in the direction of the primary flow. The second-order approximation gives a viscoelastic contribution, which depends on the geometry of the annular region as well as on all the material parameters η_0 and λ_1 to λ_7.

Appendix A

1.Table (A-1): Constitutive equations included in Oldroyd 8-constant model as special cases. Here, we make use of Oldroyd's suggestion [27,30,31] which reduces the model to 6-parameters model by assuming that $\lambda_3 = \lambda_4 = 0$

2. Approximation method

The vector $\underline{\Lambda}$ defined by Eq. (2-12), or its components in Eqs. (17), can be found through the relations:

The components of $\overset{\nabla}{\underline{\underline{T}}}$, $\overset{\nabla}{\underline{\underline{D}}}$ **and** $(tr\,\underline{\underline{T}})\underline{\underline{D}}$

(i)The tensors $\underline{\underline{D}}$ and $\underline{\underline{T}}$ have the components shown in Table (A-1)

	$\hat{r}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{Z}\hat{Z}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{Z}+\hat{Z}\hat{r}$	$\hat{\theta}\hat{Z}+\hat{Z}\hat{\theta}$
D_{ij}	$U_{,r}$	$\frac{U+V_{,\theta}}{r}$	$W_{,z}$	$\frac{U_{,\theta}+rV_{,r}-V}{2r}$	$\frac{W_{,r}+U_{,Z}}{2}$	$\frac{W_{,\theta}+rV_{,z}}{2r}$
T_{ij}	T_{rr}	$T_{\theta\theta}$	T_{ZZ}	$T_{r\theta}=T_{\theta r}$	$T_{rz}=T_{zr}$	$T_{\theta z}=T_{z\theta}$

(ii) The components of an upper-convicted tensor $\underline{\underline{A}}$ (which may be either $\underline{\underline{D}}$ or $\underline{\underline{T}}$) are given by the relations

$$\overset{\nabla}{A}_{rr} = UA_{rr,r} + \frac{V}{r}A_{rr,\theta} - 2U_{,r}A_{rr} - 2\frac{U_{,\theta}}{r}A_{r\theta}\ , \qquad \text{(A-2a)}$$

$$\overset{\nabla}{A}_{\theta\theta} = UA_{\theta\theta,r} + \frac{V}{r}A_{\theta\theta,\theta} - 2\frac{U+V_{,\theta}}{r}A_{\theta\theta} - 2\frac{V-rV_{,r}}{r}A_{r\theta}\ , \qquad \text{(A-2b)}$$

$$\overset{\nabla}{A}_{zz} = UA_{zz,r} + \frac{V}{r}A_{zz,\theta} - +2W_{,r}A_{rz} - 2\frac{W_{,\theta}}{r}A_{\theta z} \qquad \text{(A-2c)}$$

$$\overset{\nabla}{A}_{r\theta} = \overset{\nabla}{A}_{\theta r} = UA_{r\theta,r} + \frac{V}{r}A_{r\theta,\theta} - \frac{U+rU_{,r}+V_{,\theta}}{r}A_{r\theta} + \frac{V-rV_{,r}}{r}A_{rr} - \frac{U_{,\theta}}{r}A_{\theta\theta} \qquad \text{(A-2d)}$$

$$\overset{\nabla}{A}_{rz} = \overset{\nabla}{A}_{zr} = UA_{rz,r} + \frac{V}{r}A_{rz,\theta} - \frac{V+U_{,\theta}}{r}A_{\theta z} - W_{,r}A_{rr} - \frac{W_{,\theta}}{r}A_{r\theta} - U_{,r}A_{rz} \qquad \text{(A-2e)}$$

$$\overset{\nabla}{A}_{\theta z} = \overset{\nabla}{A}_{z\theta} = UA_{\theta z,r} + \frac{V}{r}A_{\theta z,\theta} - \frac{U+V_{,\theta}}{r}A_{\theta z} - \frac{W_{,\theta}}{r}A_{\theta\theta} - W_{,r}A_{\theta r} + \frac{V-rV_{,r}}{r}A_{rz} \qquad \text{(A-2f)}$$

(iii) The components of the divergence of $\overset{\nabla}{\underline{\underline{A}}}$ are:

$$(\nabla.\ \overset{\nabla}{\underline{\underline{A}}})_r = r^{-1}\partial_r(r\,\overset{\nabla}{A}_{rr}) + \frac{1}{r}\partial_\theta(\overset{\nabla}{A}_{r\theta}) - \frac{\overset{\nabla}{A}_{\theta\theta}}{r}\ , \qquad \text{(A-3a)}$$

$$(\nabla.\ \overset{\nabla}{\underline{\underline{A}}})_\theta = r^{-2}\partial_r(r^2\,\overset{\nabla}{A}_{r\theta}) + \frac{1}{r}\partial_\theta(\overset{\nabla}{A}_{\theta\theta})\ , \qquad \text{(A-3b)}$$

$$(\nabla.\ \overset{\nabla}{\underline{\underline{A}}})_z = r^{-1}\partial_r(r\,\overset{\nabla}{A}_{rz}) + \frac{1}{r}\partial_\theta(\overset{\nabla}{A}_{\theta z})\ . \qquad \text{(A-3c)}$$

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