Optimal Solution Method of Integro-Differential Equaitions under Laplace Transform

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Abstract— *In this paper, Laplace Transform method is developed to solve partial Integro-differential equations. Partial Integro-differential equations (PIDE) occur naturally in various fields of science. Engineering and Social Science. We propose a max general form of linear PIDE with a convolution Kernal. We convert the proposed PIDE to an ordinary differential equation (ODE) using the LT method. We applying inverse LT as exact solution of the problems obtained. It is observed that the LT is a simple and reliable technique for solving such equations. The proposed model illustrated by numerical examples.*

Keywords—Partial Integro-differential equations convolution Kernal, Laplace Transform.

I. INTRODUCTION

In this paper, we study the partial Integro-differential equation under the most general Laplace Transform method (LT) for linear term with a convolution kernel. Due to local nature of Ordinary differential operator, the models containing merely ODOs do not useful in modeling memory. One of the best remedies to overcome this risk factor is inclusion of the integral term in the model. The ODE and PDE along with the weighted integral of unknown function gives rise to an integro-differential equation or a partial integro-differential equation. Applications of PIDE can be found in various fields. Various numerical schemes are proposed by Dehghan [4] to solve PIDEs arising in viscosity. Appell [2] proposed a Partial Integro Operators and integro-differential equations. Non-linear PIDEs arising in nuclear reactor dynamics are solved by Pachapatte [3] on some new integral and integro-differential inequalities in two independent variables and their application and PIDEs have been used in jump-diffusion models for pricing of derivatives in finance and management. Abeerge [1] A Non-linear partial-integro-differential equations used a non-linear PIDE in financial modeling. The numerical technique basically illustrates how the Laplace Transform can be used to approximate the solution of the non-linear differential equation by manipulating the decomposition method which was first introduced by Schiff [6], the Laplace Transform Theory and applications. The most valuable method for solving linear equation is the Laplace Transform technique. Also LT is used in for calculations of water flow, wave equations, electronic circuit problems and heat transfer in fractured rocks. Merdon et al [7] proposed a revised method for non-linear oscillatory systems using LT.

In this paper we applied Laplace Transform technique of a linear PIDE under the convolution of two functions. We construct our method to approximate the solution PIDE. In section two we introduce necessary definitions of LT. In section 3, we developed the proposed method. Examples are given in section 4.

II. BASIC DEFINITIONS

2.1 Laplace Transform

Let f(t) be a real valued continuous function defined for $0 \leq t < \infty$. Suppose that for a real (or) complex parameter s, the integral $\int_0^\infty e^{-st}$ f(t) exist. Then the integral $L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)dt = F(s)$ is called Laplace Transform of f(t).

2.2 Sufficient conditions

L[f(t)] exists for $s < \infty$, if

(i) f(t) is piece wise continuous on every finite interval in the range $t \geq 0$

(ii) f(t) is of exponential order.

The above conditions are sufficient but not necessary.

2.3 LT of periodic function

If f(t) is a periodic function with period T then

$$L\left[f(t)\right] = \frac{1}{1-e^{-tT}} \int_0^T e^{-st} f(t)dt$$

2.4 Convolution Theorem

Let f(t) and g(t) be two functions, L[f(t)] = F(s); L[g(t)] = G(s) then

L-1[F(s) G(s)] = f(t) x g(t) = $\int_0^t f(u)g(t-u)du$

III. SOLUTION PROCEDURE OF PIDE

Consider PIDE,

$$\sum_{i=0}^{m} A_i \frac{\partial^i u}{\partial t^i} + \sum_{i=0}^{n} B_i \frac{\partial^i u}{\partial x^i} + Cu + \sum_{i=0}^{r} D_i \int_0^t k_i(t-s)\frac{\partial^i u(x,s)}{\partial x^i} ds + f(x,t) = 0 \qquad (1)$$

Where f(x,t) and Ki (t,s) are known functions. Ai's, Bi's and C are constants or the functions of x.

Operating Laplace Transform on both sides of (1)

$$\sum_{i=0}^{m} A_i L\left(\frac{\partial^i u}{\partial t^i}\right) + \sum_{i=0}^{n} B_i L\left(\frac{\partial^i u}{\partial x^i}\right) + CL(u)$$

$$+\sum_{i=0}^{r} D_i L\left[k_i(t-s)\frac{\partial^i u(x,t)}{\partial x^i}\right] + L\left[f(x,t)\right] = 0$$

Using convolution Theorem for LT, we get,

$$\sum_{i=0}^{m} A_i \left[s^i \bar{u}(t,s)\right] - \sum_{s=1}^{t}\left[s^{i-j}\, u^{(i-j)}(t,0)\right]$$

$$+\sum_{i=0}^{n} B_i \frac{\partial \bar{u}(t,s)}{\partial t^i} + C\,\bar{u}(t,s)$$

$$+\sum_{i=0}^{r} D_i \bar{k}_i(s)\frac{\partial^i \bar{u}(t,s)}{\partial t^i} + L\left[f(t,s)\right] = 0 \qquad (3)$$

where $\bar{u}(t,s) = L\left[u(x,t)\right]$

equation (2) is an ordinary differential equation in $\bar{u}(t,s)$.

Solving this ODE and Operating Inverse Laplace Transform of $\bar{u}(x,s)$ we get a solution u (x, t) of (1).

IV. ILLUSTRATIVE EXAMPLES

4.1 Consider the PIDE

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial y}{\partial x} + 2\int_0^t (t-s)y(x,s)ds - 2e^x \qquad (3)$$

With initial condition

y(x,0) = e-x, yi (x,0) = 0 (4)

and boundary condition

y(0,t) = Sin t (5)

Operating Laplace Transform w, r.t 't' on (3),

$$L\left(\frac{\partial^2 y}{\partial t^2}\right) = L\left(\frac{\partial y}{\partial x}\right) + 2L\left[\int_0^t (t-s)y(x,s)ds\right] - 2L(e^x)$$

$$s^2 L(y)_{(t,s)} - s\,y(t,0) - \frac{\partial y}{\partial t}(t,0)$$

$$= \frac{d\bar{y}}{dx} + 2\frac{1}{s^2}\left(\bar{y} - 2e^{-x}\right)\frac{1}{s}$$

$$= \frac{\partial \bar{y}}{\partial x} + \frac{2}{s^2}\bar{y} - 2\frac{1}{s}$$

$$\therefore \frac{\partial \bar{y}}{\partial x} + \left(\frac{2}{s^2} - s^2\right)\bar{y} = \frac{2}{s} - s \qquad (6)$$

This is a linear differential equation in $\bar{y}(u)$. The general solution of $\bar{y}$ is

Integrating factor = $\int\left(\frac{2}{s} - s\right)$. Integrating factor ds + C.

Where integrating factor $e^{\int\left(\frac{2}{s^2} - s^2\right)ds}$

$$(i.e)\ \bar{y}(t,s)\ e^{\int\left(\frac{2}{s^2} - s^2\right)ds} = \int\left(\frac{2}{s} - s\right)ds\ e^{\int\left(\frac{2}{s^2} - s^2\right)ds} dt + C$$

$$\bar{y}(t,s) = \frac{1}{s^2+1}et + Ce^{\int\left(s^2 - \frac{2}{s^2}\right)t} \qquad (7)$$

Applying the boundary conditions

$$\bar{y}(0,s) = \frac{1}{s^2+1} \qquad (8)$$

Using (7) and (8) we get C = 0

Eq (7) become,

$$\bar{y}(t,s) = \frac{1}{s^2+1}e^t \qquad (9)$$

Operating inverse Laplace Transform on (9),

$$L^{-1}\left[\bar{y}(t,s)\right] = L^{-1}\left(\frac{1}{s^2+1}e^t\right)$$

$$= e^t L^{-1}\left(\frac{1}{s^2+1}\right) = e^t \sin t$$

⇒ y(x,t) = ex Sin t

4.2 Example

Consider the PIDE

$$\frac{dy}{dt} - \frac{d^2 y}{dx^2} + y + \int_0^t e^{t-s} y(x,s)\,ds = (x^2-1)e^t - 3 \qquad (10)$$

With the initial conditions,

$$y(x,0) = x^3, \frac{\partial y}{\partial t}(x,0) = 1 \qquad (11)$$

$$y(0,t) = t, \frac{\partial y}{\partial x}(0,t) = 0 \qquad (12)$$

Now, Operating Laplace Transform of (10) w.r.t. 't'.

$$L\left(\frac{dy}{dt}\right) - L\left(\frac{d^2 y}{dx^2}\right) + L(y) + L\left(\int_0^t e^{t-s} y(x,s)\right)ds$$

$$= L\left[(x^2-1)et\right] - L(3)$$

$$sL(y) - sy(t,0) - \frac{d^2 y}{dx^2} + L(y) + \frac{1}{s-1} L(y) = \frac{1}{s-1}(x^2 - 1) - \frac{3}{s}$$

$$s\,\bar{y}(x,s) - y(x,0) - \frac{d^2\bar{y}}{dx^2} + \bar{y} + \frac{1}{s-1}\left(x^2 - 1\right) - \frac{3}{s}$$

$$\frac{d^2\bar{y}}{dx^2} - \left(\frac{s^2}{s-1}\right)\bar{y} = x^2\left(\frac{s}{s-1}\right) + \frac{s}{s-1} + \frac{3}{s}$$

This is ODE in $\bar{y}$

The auxiliary equation is $m^2 - \frac{s^2}{s-1} = 0$

$$\Rightarrow \quad m^2 = \frac{s^2}{s-1} \Rightarrow m = \pm\sqrt{\frac{s^2}{s-1}} = \pm\frac{s^2}{\sqrt{s-1}}$$

The complementary function is

$$\bar{y} = Ae^{\frac{s}{\sqrt{s-1}}x} \quad Be^{-\frac{s}{\sqrt{s-1}}x}$$

Using above and solving we get,

$$\bar{y}(x,s) = Ae^{\frac{s}{s-1}x} + Be^{-\frac{s}{s-1}x} + \frac{x^2}{s} - \frac{1}{s^2} \qquad (13)$$

Applying conditions of (11), (12),

$$y(0,t) = t \Rightarrow \bar{y}(0,s) = \frac{1}{s^2} \qquad (14)$$

and

$$\frac{\partial y}{\partial x}(0,t) = 0 \Rightarrow d\frac{\bar{y}(0,s)}{dx} = 0 \qquad (15)$$

Using (14), (15) in (13) we get

A + B = 0 (16)

A – B = 0 (17)

Solving (16), (17) we get A = 0 = B

Equation (13) becomes,

$$\bar{y}(x,s) = \frac{x^2}{s} - \frac{1}{s^2}$$

$$\Rightarrow L\left(y\right) = \frac{x^2}{s} - \frac{1}{s^2}$$

$$y = L^{-1}\left(\frac{x^2}{s} - \frac{1}{s^2}\right)$$

= x2 – t

Solution in y(x, t) = x2 – t is an exact solution.

4.3 Example consider the PIDE

$$\frac{\partial y}{\partial t} + \frac{\partial^3 y}{\partial t^3} + \frac{\partial y}{\partial t} - y + xt - \int_u^t \operatorname{Cosh}(t-s)\frac{\partial^3 y}{\partial x^3}(x,s)ds = 0 \qquad (18)$$

With the conditions,

$$y(x,0) = 0; \frac{\partial y}{\partial t}(x,0) = x, + \frac{\partial^2 y}{\partial t^2}(x,0) = 0 \qquad (19)$$

$$y(0,t) = 0; \frac{\partial y}{\partial x}(0,t) = \operatorname{Cos}\ \tau; \frac{\partial^2 y}{\partial x^2}(0,t) = 0 \qquad (20)$$

Operating Laplace Transform on (18) w.r.t. ‘t’ and using the condition (19) we get,

$$L\left(\frac{\partial y}{\partial t}\right) + L\left(\frac{\partial^3 y}{\partial t^3}\right) + L\left(\frac{\partial y}{\partial t}\right) - L(y)$$

$$+L(xt) - L\left[\int_u^t \cos t(t-s)\frac{\partial^3 y}{\partial x^3}(x,s)ds = 0\right.$$

$$sL(y) - y^1(0) + s^3 L(y) - s^2(y(0) - sy^1(0) - y^{11}(0)$$

$$-L(y) + x\frac{1}{s} - \frac{\partial^3 y}{\partial x^3} L\left[\int_u^t \cos t(t-s)\,ds = 0\right.$$

$$\frac{\partial^3\bar{y}}{\partial x^3} + \frac{1}{s^2}\left[s^3 + s^4 - s^7 - 1\right]\bar{y} = \frac{1}{s^2}(s^3 - 1)(1 - s^3) \qquad (21)$$

The auxiliary equation is,

$$m^3 + \left(s + s^2 - s^5 - \frac{1}{s^2}\right) = 0$$

Solving (21) we get,

$$\bar{y}(x,s) = Ae^{(-1)^{2/3}\frac{(1-s^3-s^4+s^7)^{1/3}x}{s^{2/3}}}$$

$$+Be^{(-1)^{1/3}\frac{(1-s^3-s^4+s^7)^{1/3}x}{s^{2/3}}} + Ce^{(1-s^3-s^4+s^7)^{1/3}x/s^{2/3}} + \frac{1}{s^2+1}x\ x$$

Using (20) we get,

A = 0; B = 0; C = 0

$$\bar{y}(x,s) = \frac{s}{s^2+1}x$$

$$L(y)[x,s] = \frac{s}{s^2+1}x$$

$$y(x,s) = L^{-1}\left(\frac{s}{s^2+1}\right)x = x\cos\ t$$

This is an exact solution.

V. CONCLUSIONS

In this article, we proposed most general linear Partial Integro-Differential Equations in modelling different applications in Engineering Problems such as wave equation, heat transfer equations under Laplace Transform technique. In LT, we used convolution Kernal theorem for finding exact solutions of 3 different examples.

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