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Inner Product Approach to Generalize the Notion of Pythagoras Theorem for Normed Spaces

Prisha Jain[1], Pratyush Singhal[2]

[1]Neerja Modi School, India
Email: prishajain.1709@gmail.com
[2]Neerja Modi School, India
Email: spratyush0707@gmail.com

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Abstract— *The Pythagorean Theorem, a fundamental result in Euclidean geometry, traditionally relates the lengths of the sides of a right-angled triangle. In this paper, we extend the classical Pythagorean Theorem into the context of normed vector spaces, using the concept of inner products. We explore how the theorem manifests in higher-dimensional spaces and provide a generalized version applicable to normed spaces beyond two dimensions. This generalization not only reinforces the geometric interpretation of the theorem but also connects it to broader mathematical frameworks such as vector spaces, norms, and inner products. The results presented here demonstrate the versatility of the Pythagorean Theorem and its relevance across various fields of mathematics, highlighting its applications in both theoretical and applied contexts.*

Keywords— ***Pythagoras Theorem, Orthogonality, Vector Spaces, Norm***

I. INTRODUCTION

The Pythagorean Theorem is one of the most remarkable and well-known results in mathematics. Historically attributed to the ancient Greek mathematician Pythagoras, this theorem states that in a right-angled triangle, the square of the length of the hypotenuse c is equal to the sum of the squares of the lengths of the other two sides, a and b. Mathematically, this is expressed as:

$$a^2 + b^2 = c^2$$

This theorem has been known and proved by various cultures throughout history. The earliest known record dates back to the Babylonians around 1800 BCE, who used it in their calculations. The Indians, particularly Baudhayana, provided a detailed proof around 800 BCE, and the Chinese also contributed with their own version in the Zhou Bi Suan Jing. The theorem was later named after Pythagoras, who lived in the 6th century BCE and is traditionally credited with its discovery, though it's likely he learned it from these earlier sources. The Pythagorean Theorem not only holds a fundamental place in Euclidean geometry but also serves as a cornerstone for various fields such as trigonometry, algebra, and even physics.

In modern mathematical language, the Pythagorean Theorem can be viewed through the lens of vector spaces and norms. Specifically, in a two-dimensional Euclidean space, the theorem is a manifestation of the inner product, where the Euclidean norm (or length) of a vector v = (a, b) is given by:

$$||v|| = \sqrt{a^2 + b^2}$$

This notion can be generalized to n-dimensional normed spaces (or vector spaces) using inner products. For a vector v = (v1, v2,…., vn) in Rn, the Euclidean norm is defined as:

$$||v|| = \sqrt{v1^2 + v2^2 + \;.\; vn^2}$$

This generalization leads to the concept of the Pythagorean Theorem in higher dimensions. For any two orthogonal vectors u and v in an inner product space, the norm of their sum satisfies:

$$||u + v||^2 = ||u||^2 + ||v||^2$$

This result is a direct extension of the classical Pythagorean Theorem.

However, the Pythagorean Theorem is specifically applicable to right-angled triangles, the reasons for which we will explore in the later sections of this paper. For non-right-angled triangles, this theorem does not hold. Instead, more general results such as the Law of Cosines are used, which account for the angle between the sides: $c^2 = a^2 + b^2 - 2ab\cos(\theta)$

where θ is the angle opposite the side c. This highlights the special nature of the right angle in the Pythagorean Theorem and underscores why the theorem is not directly applicable to non-right-angled triangles.

In this research, we explore the generalization of the Pythagorean Theorem within the framework of normed vector spaces, leveraging inner products to extend these classical concepts into broader mathematical contexts.

II. METHOD

I) VECTOR SPACES

A vector space, or linear space, is a fundamental concept in mathematics and physics, providing a framework for working with vectors. In many areas of mathematics, the concept of forming 'linear combinations' of elements within a set is both useful and significant. This idea is naturally encountered in various contexts, such as in the study of linear equations through linear combinations of matrix rows, in calculus with linear combinations of functions, and in three-dimensional Euclidean space with linear combinations of vectors. Linear algebra, as a field, focuses on the shared properties of algebraic systems characterized by a set and a coherent notion of linear combinations of its elements.

Vector Spaces is a fundamental mathematical construct that incorporates this abstraction and provides a unifying framework for analyzing and understanding these systems. By defining vector spaces and exploring their properties, we establish a foundation for numerous applications across different branches of mathematics and science. Formally, a vector space over a field F (such as the real numbers R or complex numbers C) is a set V equipped with two operations: vector addition and scalar multiplication. These operations must satisfy the following axioms for all vectors u, v, w ∈ V and scalars c, d ∈ F:

- Closure under addition: $u + v \in V$.
- Commutativity of addition: $u + v = v + u$.
- Associativity of addition: $u + (v + w) = (u + v) + w$.
- Existence of additive identity: There exists a vector 0 ∈ V such that $u + 0 = u$ for all u ∈ V
- Existence of additive inverse: For each u ∈ V, there exists a vector $u \in V$ such that $u + (-u) = 0$
- Closure under scalar multiplication: cu ∈ V
- Distributivity of scalar multiplication with respect to vector addition: $c(u + v) = cu + cv$.
- Distributivity of scalar multiplication with respect to field addition: $(c + d)u = cu + du$.
- Associativity of scalar multiplication: $c(du) = (cd)u$.
- Existence of multiplicative identity: $1(u) = u$ for all $u \in V$

These axioms provide a structure that supports many operations and concepts in linear algebra, such as linear transformations, eigenvalues, and eigenvectors. Understanding vector spaces is essential for delving into inner products and norms, which we will explore in subsequent sections.

III. INNER PRODUCT

An inner product generalizes the dot product. In a vector space, it provides a method for multiplying vectors to yield a scalar. This inner product can also be used to define the notions of 'length' and 'angle.

Definition- Let F be the field of real numbers or the field of complex numbers, and V a vector space over F . An inner product on V is a function which assigns to each ordered pair of vectors x, y ∈ V a scalar ⟨x, y⟩ ∈ F in such a way that for all x, y, z ∈ V and all

scalars β, the following properties hold:

(a) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,

(b) $\langle \beta x, y\rangle = \beta\langle x, y\rangle$,

(c) $\langle y, x\rangle = \overline{\langle x, y\rangle}$, the bar denoting complex conjugation,

(d) ⟨x, x⟩ > 0 if x != 0.

The vector space V with an inner product is called a (real) inner product space.

III) NORMS

By the third axiom ⟨u, u⟩ ≥ 0 of an inner product, ⟨u, u⟩ is nonnegative for any vector u. Thus, its positive square root exists. We use the notation

$\|u\| = p\langle u, u\rangle$

This nonnegative number is called the norm or length of u. The relation $\|u\|^2 = \langle u, u\rangle$ will is used frequently.

Definition- A vector norm is a function from Rn to R, with a certain number of properties. If $x \in R^n$, we symbolize its norm by ‖x‖. The defining properties of a norm are: 1. ‖x‖ ≥ 0 for all x ∈ Rn and also ‖x‖ = 0 if and only

if x=0.

2. $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $\alpha \in R$ and $x \in R^n$.

3. $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in R^n$.

IV. ORTHOGONALITY

Let V be an inner product space. The vectors $x, y \in V$ are said to be orthogonal and x is said to be orthogonal to y if $\langle x, y\rangle = 0$

The relation is clearly symmetric—if x is orthogonal to y, then $\langle y, x\rangle = 0$, and so y is orthogonal to x. We note that 0 $\in V$ is orthogonal to every $y \in V$, because

$$\langle 0, y\rangle = \langle 0y, y\rangle = 0\ \langle y, y\rangle = 0$$

Conversely, if x is orthogonal to every $y \in V$, then $\langle x, x\rangle = 0$ by [I3]. Observe that x and v are orthogonal if and only if cos α =0, where α is the angle between x and y. Also, this is true if and only if x and y are "perpendicular"- that is,

$\alpha = \frac{\pi}{2}$ (or $\alpha = 90^{\circ}$)

V. RESULTS

I) GENERALISED PYTHAGORAS THEOREM

Let $u, v \in V$ and u is orthogonal to v then

$\| u + v||^2 = \| u \|^2 + \|v\|^2$

Proof-

If $\|u + v\|^2 = \langle u + v, u + v\rangle$

Then

$+ \langle u, v\rangle + \langle v, \rangle + \langle v, u\rangle +$

$\|u + v\|^2 = \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle =$

$||u||^2$

$\| v \|^2$

$= \| u \|^2 + \|v\|^2$

For n-dimensions-

$$|| \sum_{i=1}^{n} u_i||^2 = || \sum_{i=1}^{n} u_i||^2$$

$$||u_1 + u_2 + \ldots\ldots u_n||^2 = ||u_1||^2 + ||u_2||^2 + \ldots\ldots + ||u_n||^2$$

VI. CONCLUSION

In conclusion, this paper has successfully extended the classical Pythagorean Theorem, traditionally applicable to right-angled triangles in Euclidean geometry, into a more generalized framework suitable for normed vector spaces. By leveraging the concepts of inner products and norms, we have demonstrated that the core principles of the Pythagorean Theorem can be applied within higher-dimensional spaces, allowing for broader mathematical and practical applications. This generalization provides a foundational understanding that bridges classical geometry with more advanced linear algebra, offering insights that can be utilized in various fields of mathematics and science. Through this exploration, the intrinsic relationship between vector norms and inner products has been elucidated, reinforcing the theorem's importance and versatility in modern mathematical contexts.

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